NASA TN D-1854

NASA TN D-1854

28 p

N63 18888

Code 1

# TECHNICAL NOTE

## D-1854

# A THEORETICAL BASIS FOR MECHANICAL IMPEDANCE SIMULATION IN SHOCK AND VIBRATION TESTING OF ONE-DIMENSIONAL SYSTEMS

F. J. On and R. O. Belsheim

Goddard Space Flight Center 603/307

Greenbelt, Maryland

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

WASHINGTON August 1963

# A THEORETICAL BASIS FOR MECHANICAL IMPEDANCE SIMULATION IN SHOCK AND VIBRATION TESTING OF ONE-DIMENSIONAL SYSTEMS

by

F. J. On and R. O. Belsheim

*Goddard Space Flight Center*

SUMMARY

18888

The feasibility of mechanical impedance control and simulation in environmental shock and vibration testing of mechanical structures was investigated theoretically. Fundamental and useful mathematical expressions for impedance are given for the analysis, control, and simulation in mechanical structures that can be considered as one-dimensional linear-passive systems. The validation of these expressions in the application to impedance control and simulation is approached primarily from analytical considerations. General theories of application are summarized, along with a few simple examples of specific cases. The overall results of the investigation, as well as certain conclusions that may be drawn, are discussed. Several recommendations are made in connection with the information obtained from this study.

## CONTENTS

# A THEORETICAL BASIS FOR MECHANICAL IMPEDANCE SIMULATION IN SHOCK AND VIBRATION TESTING OF ONE-DIMENSIONAL SYSTEMS

by

F. J. On and R. O. Belsheim*

*Goddard Space Flight Center*

## INTRODUCTION

One of the most vexing problems confronting the environmental test engineer today is the problem of how to achieve a more realistic simulation of the shock and vibration environment in the test laboratory. The shock and vibration environmental testing of today attempts to incorporate a certain degree of conservatism in its techniques and procedures for defining a qualification test. It is necessary for the test engineer to consider the problem of how to minimize such conservatism so that a higher degree of *optimum* design criteria can be approached. The intent of this paper is to present methods, based on the concepts of mechanical impedance, whereby simulation in environmental testing techniques and procedures may be optimized.

It is well known that the analysis of structures under dynamic loadings requires that the dynamic characteristics of the structures be known in addition to the characteristics of the excitations. Based on usage it is less well known that the *simulated* dynamic environmental test in the laboratory should depend similarly on these requirements. A useful method of specifying these requirements, for structures that can be considered as linear elastic systems, is in terms of mechanical impedance. The concepts of mechanical impedance are directly analogous to the concepts of electrical impedance, and many of the same theorems and operational methods apply (Reference 1). Although the concepts of mechanical impedance have been employed for many years in the analysis of dynamic responses of structures (References 2, 3, and 4), little contribution has been made toward a solution of the *simulation* and *control* of pertinent structural dynamic characteristics in the test laboratory. The importance of attempting such a solution cannot be overemphasized. As the problem of achieving optimum structural design criteria becomes increasingly important, particularly in the areas of aerospace applications, the need for a solution is readily evident.

*Dr. Belsheim, Head of the Structures Branch, Mechanics Division, Naval Research Laboratory, Washington 25, D.C., served as a consultant to Goddard Space Flight Center for the study reported herein.

# IMPEDANCE ANALYSES OF ONE-DIMENSIONAL SYSTEMS

Continuous mechanical systems that can be considered as linear systems often can be evaluated by applying the concepts of mechanical impedance. The adequacy of such an approach is generally dependent on the exactness of the assumption that the system is linear and *passive.** Consequently, in the analyses that follow, this assumption is tacitly made. Furthermore, for convenience in presentation the following analyses are referred to aerospace-vehicle - payload and testing-machine - payload systems, which are considered the most significant system combinations because weight is so important.

## Theoretical Relations

An assumption that any realistic aerospace structure behaves as a one-dimensional system is unquestionably an oversimplification. The prevalence of modes of vibration due to bending, shear, and rotary inertia forces is frequently observed in the field. However, to a first approximation, significant information can often be obtained from a one-dimensional consideration; and this can be of great advantage to the design, analysis, and qualification of the structure.

A general representation of a vehicle-payload system, which is identified by $V_a$ and $P_b$ for the vehicle and payload respectively, is depicted in Figure 1. For the sake of analysis, the configuration of the vehicle-payload has been approximated by the method of lumped parameters. By the application of the force-current analogy for dynamical systems, the vehicle-payload representation of Figure 1 is further simplified by its equivalent mechanical network, as shown in Figure 2. The application of the mechanical network theory (Norton's theorem) further reduces the network of Figure 2 to that of Figure 3. In Figure 3, $Q_{V_a}^b(s)$ is termed the transform of the force $F(t)$ measured at terminal $a$ when that terminal is blocked so that no motion occurs (hereafter, $Q_{V_a}^b(s)$ shall be termed *blocked force*); the impedances $Z_{V_a}(s)$ and $Z_{P_b}(s)$ are the transform mechanical impedance of the vehicle and payload looking back from terminal $a$, respectively. The quantity $\dot{q}_{V_a 0}(s)$(hereafter termed *free velocity*) is the transform of the velocity $\dot{x}(t)$ measured at terminal $a$ when that terminal is unrestrained. In general, these transform quantities are complex.

### *Steady-State Vibrations*

In the case of steady-state sinusoidal vibrations, the transform velocity at terminal $a$ in Figure 3 prior to the connection of the payload impedance $Z_{P_b}(s)$ is given by

$$\dot{q}_{V_a 0}(s) = Z_{V_a}^{-1}(s)\, Q_{V_a}^b(s) \; . \tag{1}$$

**Passive* systems as defined here are systems that do not possess any type of internal energy sources.

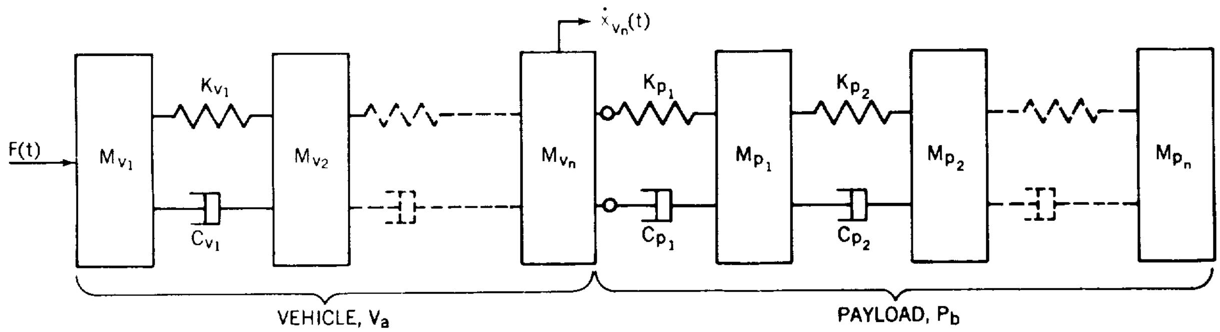



Figure 1—Lumped parameter representation of one-dimensional vehicle-payload system.

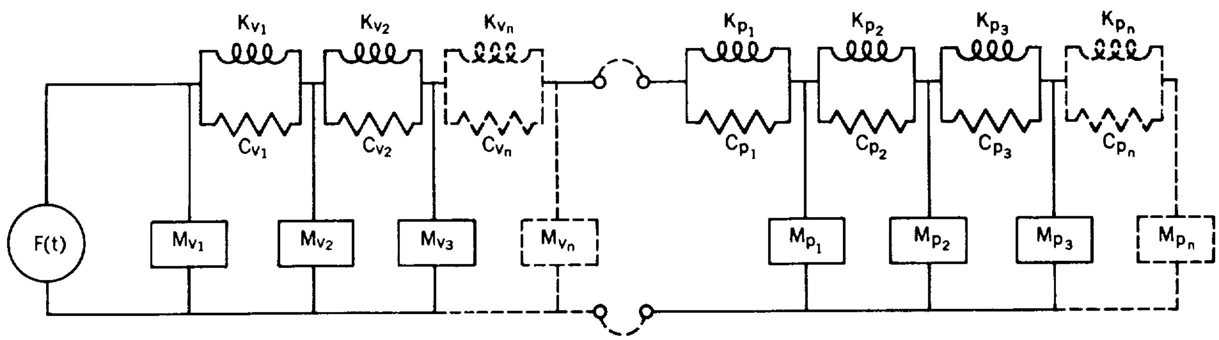



Figure 2—Mechanical network equivalent of Figure 1.

(All symbols are defined in Appendix A.) With the mechanical impedance $Z_{P_b}(s)$ of the payload connected to terminal a, the resultant mechanical impedance is

$$Z_{V_{ab}}(s) = Z_{V_a}(s) + Z_{P_b}(s), \tag{2}$$

and the transform velocity at the joining terminal of a and b becomes

$$\dot{q}_{V_{ab}}(s) = H_1(s)\, Q_{V_a}^{b}(s), \tag{3}$$

where

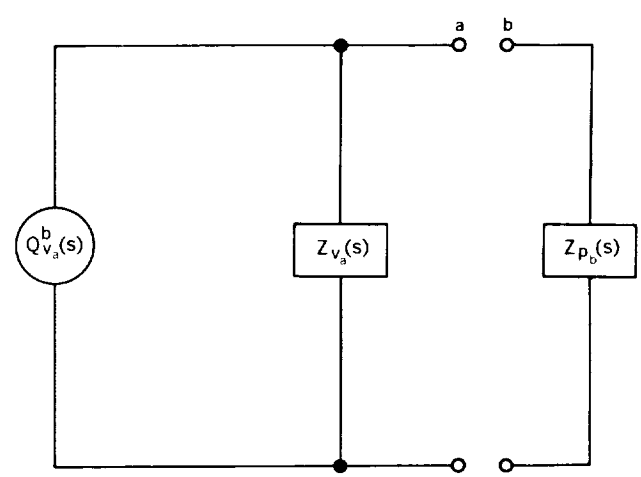



Figure 3—Simplified mechanical network equivalent of a vehicle-payload system by use of Norton's theorem.

$$H_1(s) = \left[Z_{V_a}(s) + Z_{P_b}(s)\right]^{-1} \tag{4}$$

may be considered as some frequency response function. Since Equation 1 shows that

$$Q_{V_a}^{b}(s) = Z_{V_a}(s)\,\dot{q}_{V_{a}0}(s)\,, \tag{5}$$

then Equation 3, in terms of *free velocity* $\dot{q}_{V_{a}0}(s)$, becomes

$$\dot{q}_{V_{ab}}(s) = H_2(s)\,\dot{q}_{V_{a}0}(s)\,, \tag{6}$$

where

$$H_2(s) = Z_{V_a}(s)\left[Z_{V_a}(s) + Z_{P_b}(s)\right]^{-1} \tag{7}$$

may be considered as another frequency response function.

With $H_1(s)$ and $H_2(s)$ predetermined, Equation 3 or 6 may be used to predict the resulting velocity characteristic at the terminal joining the vehicle $V_a$ and payload $P_b$.

In cases whereby either $V_a$ or $P_b$ is replaced by another $V_k$ or $P_r$, new velocity characteristics may be predicted. Suppose that the velocity characteristic corresponding to a new payload $P_r$ on the same vehicle $V_a$ is desired; then, using Equations 3 and 6, the new velocity is

$$\dot{q}_{V_{ar}}(s) = H_3(s)\,\dot{q}_{V_{ab}}(s)\,, \tag{8}$$

where

$$H_3(s) = \left[Z_{V_a}(s) + Z_{P_b}(s)\right]\left[Z_{V_a}(s) + Z_{P_r}(s)\right]^{-1} \tag{9}$$

and $Z_{P_r}(s)$ is the transform mechanical impedance of the new payload $P_r$.

Similarly, for the same payload $P_b$ on a new vehicle $V_k$, the new velocity may be predicted by*

$$\dot{q}_{V_{kb}}(s) = H_4(s)\,\dot{q}_{V_{ab}}(s), \tag{10}$$

where

$$H_4(s) \quad \left[\dot{q}_{V_k0}(s)\right]\left[\dot{q}_{V_a0}(s)\right]^{-1}\left[\frac{Z_{V_a}(s)+Z_{P_b}(s)}{Z_{V_a}(s)}\right]\left[\frac{Z_{V_k}(s)+Z_{P_b}(s)}{Z_{V_k}(s)}\right]^{-1}; \tag{11}$$

or by

$$\dot{q}_{V_{kb}}(s) = H_5(s)\,\dot{q}_{V_{ab}}(s), \tag{12}$$

in which

$$H_5(s) = \left[Q^b_{V_k}(s)\right]\left[Q^b_{V_a}(s)\right]^{-1}\left[Z_{V_a}(s)+Z_{P_b}(s)\right]\left[Z_{V_k}(s)+Z_{P_b}(s)\right]^{-1}. \tag{13}$$

It follows that other variations of Equations 3 and 6 may be accomplished. One that seems most significant is the case of a payload $P_b$ designed to ride on a vehicle $V_a$, but to be vibration-tested on a machine of dynamic characteristic $Z_{S_m}(s)$. To provide a realistic test, the dynamic characteristic of $V_a$ should be simulated by the test machine. By the method of lumped parameters, the simplified vehicle-payload and machine-payload systems may be represented by Figures 3 and 4, respectively. From Equations 3 and 6, we may obtain (with appropriate subscripts) the following:

$$\dot{q}_{V_{ab}}(s) = H_2(s)\,\dot{q}_{V_a0}(s), \tag{14}$$

$$\dot{q}_{S_{mb}}(s) = H_6(s)\,\dot{q}_{S_m0}(s), \tag{15}$$

where

$$H_6(s) = Z_{S_m}(s)\left[Z_{S_m}(s)+Z_{P_b}(s)\right]^{-1}; \tag{16}$$

*This prediction, based on Equations 10 to 13, is only of academic interest, since all terms in Equations 3 to 7 are required; it may be done with the simpler equations.

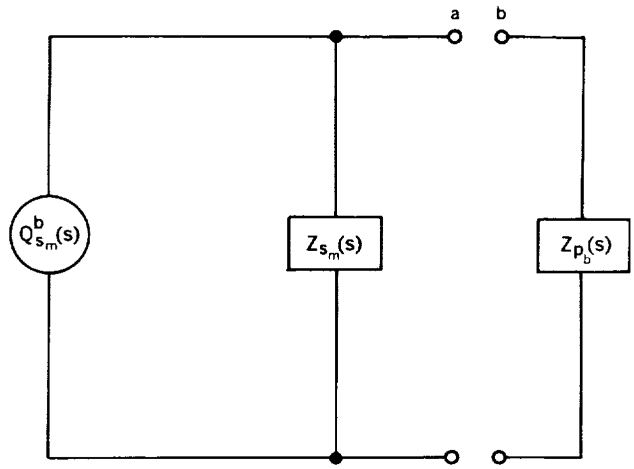



Figure 4—Simplified mechanical network equivalent of a test-machine - payload system by use of Norton's theorem.

or, in terms of *blocked force,*

$$\dot{q}_{v_{ab}}(s) = H_1(s)\, Q^b_{v_a}(s)\,, \tag{17}$$

$$\dot{q}_{s_{mb}}(s) = H_7(s)\, Q^b_{s_m}(s)\,, \tag{18}$$

where

$$H_7(s) = \left[Z_{s_m}(s) + Z_{p_b}(s)\right]^{-1}. \tag{19}$$

For the condition of simulation

$$\dot{q}_{s_{mb}}(s) = \dot{q}_{v_{ab}}(s)\,; \tag{20}$$

and this requires that

$$\dot{q}_{s_{m0}}(s) = E_{2,6}(s)\, \dot{q}_{v_{a0}}(s)\,, \tag{21}$$

where

$$E_{2,6}(s) = H_2(s)\, H_6^{-1}(s)\,; \tag{22}$$

or, in terms of *blocked force,*

$$\dot{q}_{s_{m0}}(s) = E_{1,6}(s)\, Q^b_{v_a}(s)\,, \tag{23}$$

where

$$E_{1,6}(s) = H_1(s)\, H_6^{-1}(s)\,. \tag{24}$$

With $\dot{q}_{s_{m0}}(s)$ from Equation 21 or 23, the dynamic characteristic of the vehicle-payload system of Figure 3 is simulated in the system of Figure 4, in which a test machine has replaced the vehicle; that is,

$$\dot{q}_{s_{mb}}(s) \rightarrow \dot{q}_{v_{ab}}(s) = H_2(s)\, \dot{q}_{v_{a0}}(s) \tag{25}$$

or

$$\dot{q}_{s_{mb}}(s) \rightarrow \dot{q}_{v_{ab}}(s) = H_1(s)\, Q^b_{v_a}(s)\,. \tag{26}$$

Equations 21 and 23 yield relations between the *free velocity* characteristics of the machine and the *free velocity* or *blocked force* characteristics at the terminal of the vehicle. As a consequence of Equations 21 and 23, Equations 25 and 26 are obtained. These are statements of simulation for one-dimensional systems that are linear and passive.

So far, the steady-state case in which the forcing function is a harmonic function of time has been considered. If the forcing function is given by a series of harmonic functions, the response can be obtained by superposition of the elementary solutions. The principle of superposition can be applied to all cases in which the effect of simultaneous superposed actions is the sum of the effects of each individual action. Thus, in the general case of steady-state vibrations, the principle of superposition yields for any periodic excitation the equations analogous to Equations 1 to 26.

If it is assumed that the *blocked force* is given by a series of the form

$$F^b(t) = \sum_{N=1}^{M} F_N^b \, e^{iN\omega t} \tag{27}$$

and the *free velocity* by

$$\dot{x}_0(t) = \sum_{N=1}^{M} \dot{x}_{0,N} \, e^{iN\omega t} \qquad (M = 1, 2, \cdots), \tag{28}$$

then the equations analogous to Equations 3 and 6 are respectively:

$$\dot{x}_{V_{ab}}(t) = \sum_{N=1}^{M} H_1(iN\omega) \, F_{V_a,N}^b \, e^{iN\omega t} \qquad (M = 1, 2, \cdots), \tag{29}$$

where

$$H_1(iN\omega) = H_1(s)\Big|_{s=iN\omega}; \tag{30}$$

and

$$\dot{x}_{V_{ab}}(t) = \sum_{N=1}^{M} H_2(iN\omega) \, \dot{x}_{V_a 0,N} \, e^{iN\omega t}, \tag{31}$$

where

$$H_2(iN\omega) = H_2(s)\Big|_{s=iN\omega}. \tag{32}$$

Analogous to Equation 8,

$$\dot{x}_{v_{ar}}(t) = \sum_{N=1}^{M} H_3(iN\omega)\, \dot{x}_{v_{ab},N}\, e^{iN\omega t} , \tag{33}$$

where

$$H_3(iN\omega) = H_3(s)\big|_{s=iN\omega} . \tag{34}$$

Analogous to Equations 10 and 12 respectively:

$$\dot{x}_{v_{kb}}(t) = \sum_{N=1}^{M} H_4(iN\omega)\, \dot{x}_{v_{ab},N}\, e^{iN\omega t} , \tag{35}$$

where

$$H_4(iN\omega) = H_4(s)\big|_{s=iN\omega} ; \tag{36}$$

and

$$\dot{x}_{v_{kb}}(t) = \sum_{N=1}^{M} H_5(iN\omega)\, \dot{x}_{v_{ab},N}\, e^{iN\omega t} , \tag{37}$$

where

$$H_5(iN\omega) = H_5(s)\big|_{s=iN\omega} . \tag{38}$$

Analogous to Equations 21 and 23, respectively:

$$\dot{x}_{s_m 0}(t) = \sum_{N=1}^{M} E_{2,6}(iN\omega)\, \dot{x}_{v_a 0,N}\, e^{iN\omega t} , \tag{39}$$

where

$$E_{2,6}(iN\omega) = E_{2,6}(s)\big|_{s=iN\omega} ; \tag{40}$$

and

$$\dot{x}_{S_{m0}}(t) = \sum_{N=1}^{M} E_{1,6}(iN\omega) F^{b}_{V_a,N} e^{iN\omega t} , \tag{41}$$

where

$$E_{1,6}(iN\omega) = E_{1,6}(s)\Big|_{s=iN\omega} . \tag{42}$$

As a consequence of Equations 39 and 41, the statements of simulation for the general case of periodic excitations, which are analogous to Equations 25 and 26, are respectively

$$\dot{x}_{S_{mb}}(t) \rightarrow \dot{x}_{V_{ab}}(t) = \sum_{N=1}^{M} H_2(iN\omega)\, \dot{x}_{V_{a0,N}} e^{iN\omega t} \tag{43}$$

and

$$\dot{x}_{S_{mb}}(t) \rightarrow \dot{x}_{V_{ab}}(t) = \sum_{N=1}^{M} H_1(iN\omega)\, F^{b}_{V_a,N} e^{iN\omega t} . \tag{44}$$

*Transient Motions*

From the literature (Reference 2) transient force $F(t)$ and velocity $\dot{x}(t)$ may be expressed by the Fourier integral, while the inverse relation also holds. The response $\dot{x}(t)$ to the transient force $F(t)$ is related to the mechanical impedance $Z(i\omega)$ of the system, and the Fourier spectrum $G_F(i\omega)$ of the transient force $F(t)$. The Fourier spectrum of $\dot{x}(t)$ is

$$G_{\dot{x}}(i\omega) = Z^{-1}(i\omega)\, G_F(i\omega) \tag{45}$$

and the response itself is

$$\dot{x}(t) = \int_{\omega=-\infty}^{+\infty} Z^{-1}(i\omega)\, G_F(i\omega)\, e^{i\omega t}\, d\omega . \tag{46}$$

From Equation 45, mechanical impedance may be expressed by

$$Z(i\omega) = G_{\dot{x}}^{-1}(i\omega)\, G_F(i\omega) . \tag{47}$$

It follows from Equation 45, denoting with appropriate subscripts, that the equations for transient motions corresponding to the steady-state Equations 3 and 6 are respectively

$$G_{\dot{x}_{V_{ab}}}(i\omega) = H_1(i\omega)\, G_{F^b_{V_a}}(i\omega) \tag{48}$$

and

$$G_{\dot{x}_{V_{ab}}}(i\omega) = H_2(i\omega)\, G_{\dot{x}_{V_a 0}}(i\omega) \ ; \tag{49}$$

and the response itself is

$$\dot{x}_{V_{ab}}(t) = \int_{\omega=-\infty}^{+\infty} H_1(i\omega)\, G_{F^b_{V_a}}(i\omega)\, e^{i\omega t} d\omega \tag{50}$$

or

$$\dot{x}_{V_{ab}}(t) = \int_{\omega=-\infty}^{+\infty} H_2(i\omega)\, G_{\dot{x}_{V_a 0}}(i\omega)\, e^{i\omega t} d\omega \ . \tag{51}$$

In a similar manner the equations for the transient motions that are analogous to the equations for the steady state are obtained:

From Equation 8,

$$\dot{x}_{V_{ar}}(t) = \int_{\omega=-\infty}^{+\infty} H_3(i\omega)\, G_{\dot{x}_{V_{ab}}}(i\omega)\, e^{i\omega t} d\omega \ . \tag{52}$$

From Equations 10 and 12,

$$\dot{x}_{V_{kb}}(t) = \int_{\omega=-\infty}^{+\infty} \left[ E_{8,2}(i\omega)\, G^{-1}_{\dot{x}_{V_a 0}}(i\omega)\, G_{\dot{x}_{V_k 0}}(i\omega) \right] G_{\dot{x}_{V_{ab}}}(i\omega)\, e^{i\omega t} d\omega \ , \tag{53}$$

where

$$E_{8,2}(i\omega) = H_2^{-1}(i\omega)\, H_8(i\omega) \ ; \tag{54}$$

or

$$\dot{x}_{V_{kb}}(t) = \int_{\omega=-\infty}^{+\infty} \left[ E_{9,1}(i\omega)\, G^{-1}_{F^b_{V_a}}(i\omega)\, G_{F^b_{V_k}}(i\omega) \right] G_{\dot{x}_{V_{ab}}}(i\omega)\, e^{i\omega t} d\omega \ , \tag{55}$$

in which

$$E_{9,1}(i\omega) = H_1^{-1}(i\omega) H_9(i\omega) \cdot \tag{56}$$

From Equations 21 and 23,

$$\dot{x}_{S_m 0}(t) = \int_{\omega=-\infty}^{+\infty} E_{2,6}(i\omega) G_{\dot{x}_{V_a 0}}(i\omega) e^{i\omega t} d\omega \tag{57}$$

or

$$\dot{x}_{S_m 0}(t) = \int_{\omega=-\infty}^{+\infty} E_{1,6}(i\omega) G_{F_{V_a}^b}(i\omega) e^{i\omega t} d\omega . \tag{58}$$

From Equations 25 and 26,

$$\dot{x}_{S_{mb}}(t) - \dot{x}_{V_{ab}}(t) = \int_{\omega=-\infty}^{+\infty} H_2(i\omega) G_{\dot{x}_{V_a 0}}(i\omega) e^{i\omega t} d\omega \tag{59}$$

or

$$\dot{x}_{S_{mb}}(t) - \dot{x}_{V_{ab}}(t) = \int_{\omega=-\infty}^{+\infty} H_1(i\omega) G_{F_{V_a}^b}(i\omega) e^{i\omega t} d\omega . \tag{60}$$

Equations 48 to 60 may be used for predicting responses due to transient excitations in one-dimensional systems that are linear and passive.

*Random Vibrations*

In the case of random vibrations (Reference 5), if $S_F(\omega)$ is the spectral density of a *stationary ergodic** random force (with a Gaussian probability distribution) applied at a point, the spectral density of the response at the same point is

$$S_{\dot{x}}(\omega) = \left| Z^{-1}(i\omega) \right|^2 S_F(\omega) , \tag{61}$$

*A *stationary* random variable may be crudely defined as one whose source does not change character. An *ergodic* variable is one that obeys the "ergodic hypothesis" of statistical mechanics; this hypothesis permits consideration of an event that occurs some percentage of total time as an event with a corresponding probability of occurrence.

where $|Z^{-1}(i\omega)|^2$ is the square of the absolute value of either the reciprocal complex impedance function or the mobility (inverse of impedance). The root-mean-square (rms) value of the motion is

$$\dot{x}(\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty}\left|Z^{-1}(i\omega)\right|^2 S_F(\omega)\, d\omega\right]^{1/2} . \tag{62}$$

It follows from the fundamental relation, Equation 61 (after denoting with appropriate subscripts), that the equations for random vibrations which are analogous to those of the steady-state vibrations may be obtained.

With Equation 61, the equations for random vibrations corresponding to the steady-state Equations 3 and 6 are, respectively,

$$S_{\dot{x}_{V_{ab}}}(\omega) = \left|H_1(i\omega)\right|^2 S_{F^b_{V_a}}(\omega) , \tag{63}$$

$$S_{\dot{x}_{V_{ab}}}(\omega) = \left|H_2(i\omega)\right|^2 S_{\dot{x}_{V_{a0}}}(\omega) . \tag{64}$$

The rms value of $\dot{x}_{V_{ab}}$ may be obtained from Equations 63 or 64:

$$\dot{x}_{V_{ab}}(\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty}\left|H_1(i\omega)\right|^2 S_{F^b_{V_a}}(\omega)\, d\omega\right]^{1/2} , \tag{65}$$

or

$$\dot{x}_{V_{ab}}(\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty}\left|H_2(i\omega)\right|^2 S_{\dot{x}_{V_{a0}}}(\omega)\, d\omega\right]^{1/2} . \tag{66}$$

Likewise, analogous to the steady-state Equation 8,

$$S_{\dot{x}_{V_{ar}}}(\omega) = \left|H_3(i\omega)\right|^2 S_{\dot{x}_{V_{ab}}}(\omega) , \tag{67}$$

$$\dot{x}_{V_{ar}}(\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty}\left|H_3(i\omega)\right|^2 S_{\dot{x}_{V_{ab}}}(\omega)\, d\omega\right]^{1/2} . \tag{68}$$

Analogous to Equations 10 and 12, respectively,

$$S_{\dot{x}_{V_{kb}}}(\omega) = \left|E_{8,2}(i\omega)\right|^2 S^{-1}_{\dot{x}_{V_{a0}}}(\omega)\, S_{\dot{x}_{V_{k0}}}(\omega)\, S_{\dot{x}_{V_{ab}}}(\omega) , \tag{69}$$

$$S_{\dot{x}_{V_{kb}}}(\omega) = \left|E_{9,1}(i\omega)\right|^2 S^{-1}_{F^b_{V_a}}(\omega)\, S_{F^b_{V_k}}(\omega)\, S_{\dot{x}_{V_{ab}}}(\omega) ; \tag{70}$$

and their rms values are

$$\dot{x}_{V_{kb}}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|E_{8,2}\ (i\omega)\right|^2 S^{-1}_{\dot{x}_{V_a 0}}\ (\omega)\ S_{\dot{x}_{V_k 0}}\ (\omega)\ S_{\dot{x}_{V_{ab}}}\ (\omega)\ d\omega\right]^{1/2} , \tag{71}$$

$$\dot{x}_{V_{kb}}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|E_{9,1}\ (i\omega)\right|^2 S^{-1}_{F^b_{V_a}}\ (\omega)\ S_{F^b_{V_k}}\ (\omega)\ S_{\dot{x}_{V_{ab}}}\ (\omega)\ d\omega\right]^{1/2} . \tag{72}$$

Analogous to Equations 21 and 23, respectively,

$$S_{\dot{x}_{S_m 0}}(\omega) = \left|E_{2,6}\ (i\omega)\right|^2 S_{\dot{x}_{V_a 0}}\ (\omega) , \tag{73}$$

$$S_{\dot{x}_{S_m 0}}(\omega) = \left|E_{1,6}\ (i\omega)\right|^2 S_{F^b_{V_a}}\ (\omega) ; \tag{74}$$

and their rms values are

$$\dot{x}_{S_m 0}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|E_{2,6}\ (i\omega)\right|^2 S_{\dot{x}_{V_a 0}}(\omega)\ d\omega\right]^{1/2} , \tag{75}$$

$$\dot{x}_{S_m 0}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|E_{1,6}\ (i\omega)\right|^2 S_{F^b_{V_a}}\ (\omega)\ d\omega\right]^{1/2} . \tag{76}$$

Analogous to Equations 25 and 26, respectively,

$$S_{\dot{x}_{S_{mb}}}\ (\omega) \rightarrow S_{\dot{x}_{V_{ab}}}\ (\omega) = \left|H_2\ (i\omega)\right|^2 S_{\dot{x}_{V_a 0}}\ (\omega) , \tag{77}$$

$$S_{\dot{x}_{S_{mb}}}\ (\omega) \rightarrow S_{\dot{x}_{V_{ab}}}\ (\omega) = \left|H_1\ (i\omega)\right|^2 S_{F^b_{V_a}}\ (\omega) ; \tag{78}$$

and their rms values are

$$\dot{x}_{S_{mb}}\ (\mathrm{rms}) \rightarrow \dot{x}_{V_{ab}}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|H_2\ (i\omega)\right|^2 S_{\dot{x}_{V_a 0}}\ (\omega)\ d\omega\right]^{1/2} , \tag{79}$$

$$\dot{x}_{S_{mb}}\ (\mathrm{rms}) \rightarrow \dot{x}_{V_{ab}}\ (\mathrm{rms}) = \left[\frac{1}{2\pi}\int_0^{\infty} \left|H_1\ (i\omega)\right|^2 S_{F^b_{V_a}}\ (\omega)\ d\omega\right]^{1/2} . \tag{80}$$

Equations 63 to 80 may be used for predicting responses due to random excitations in one-dimensional systems that are linear and passive.

## THEORIES OF APPLICATION

The equations of the preceding section show how the effects of mechanical impedance variations on motion may be determined. Although these determinations are generally quite involved, it is recommended that some applications (calculations or measurements) of this type be made on actual systems before their dynamic environments are firmly specified. With the dynamic environment defined for one set of systems, the dynamic environment for another set may be predicted by using the appropriate equations obtained in the last section. After the dynamic environment is defined, the next obvious step is to attempt to simulate it in the testing environment. In the following sections, various methods of application that may be useful in achieving some degree of "simulation" are presented.

### Development of Specifications

The development of realistic shock and vibration specifications has been hindered largely by the fact that the true dynamic environmental conditions have not been known until the system actually has undergone service environments. Consequently, attempts to consider new dynamic environments often result in merely modifying existing specifications into new specifications to be used for testing *prior* to subjecting the system to service. Unless specifications are changed and are developed on a sound basis, they may be much in error. It is evident from the relations previously given that the dynamic characteristics of the system components should be considered. These equations thus provide a set of relations by use of which realistic specifications may be rectified or formulated. Simple examples illustrating the application of the relations are given in the following examples.

*Examples of Application*

Simple examples illustrating the application of Equations 3, 29, 50, and 63 are given in the following discussion.

Measurement Approach

Consider a block representation of a foundation-equipment system as depicted in Figure 5. In Figure 5b, let $F_b(t) = 0$. We wish to predict the velocity at terminal b resulting from the application of $F_a(t)$ at terminal a, after connecting an equipment of mechanical impedance $Z_e$ to the terminal b.

First, block the terminal b and measure the blocked force $F_b^b(t)$ exerted at that terminal; the quantity $F_a(t)$ is the normal exciting forcing function acting. Its nature—that is, whether it is steady-state, transient, or random—determines which cases below must be used. Replace all force generators with impedances equal to their internal impedances; then measure the mechanical impedance $Z_f$ looking back into the foundation from the terminal b, and the mechanical impedance $Z_e$ looking into the equipment from the terminal b.

1. <u>Steady-state case:</u>

From Equation 3, the velocity *phasor** $\dot{q}_b$ corresponding to the velocity $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

$$\dot{q}_b = \left[Z_f + Z_e\right]^{-1} Q_b^b , \qquad (a)$$

where $Q_b^b$ is the force phasor corresponding to the blocked force $F_b^b(t)$.

2. <u>Transient case:</u>

From Equation 50, the response $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

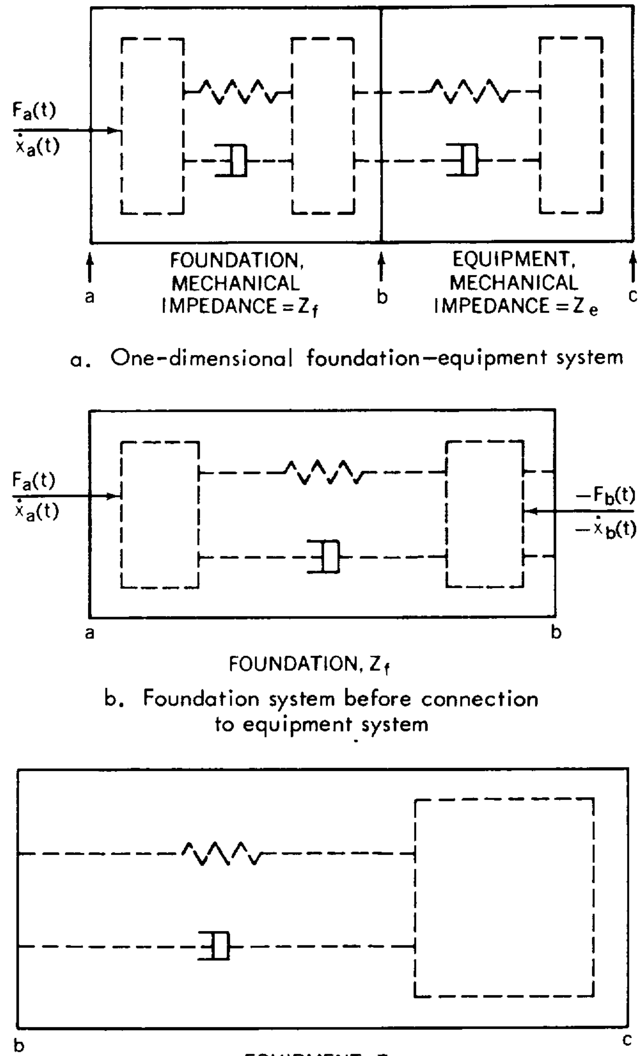
$$\dot{x}_b(t) = \int_{-\infty}^{+\infty} \left[Z_f(i\omega) + Z_e(i\omega)\right]^{-1} G_{F_b^b}(i\omega)\, e^{i\omega t}\, d\omega , \qquad (b)$$

where $G_{F_b^b}(i\omega)$ is the Fourier spectrum of the blocked force $F_b^b(t)$.

3. <u>Random case:</u>

From Equation 63, the spectral density $S_{\dot{x}_b}(\omega)$ of the response $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

$$S_{\dot{x}_b}(\omega) = \left|Z_f(i\omega) + Z_e(i\omega)\right|^{-2} S_{F_b^b}(\omega) , \qquad (c)$$




a. One-dimensional foundation–equipment system

b. Foundation system before connection to equipment system

c. Equipment system before connection to foundation system

Figure 5—Block diagrams of foundation and equipment systems.

where $S_{F_b^b}(\omega)$ is the spectral density of the blocked force $F_b^b(t)$.

## Analytical Approach

Consider a lumped parameter representation of a foundation-equipment system as depicted in Figure 6. Replace this representation by its equivalent mechanical network shown in Figure 7.

*The term *phasor* as used here implies magnitude and phase of a quantity.

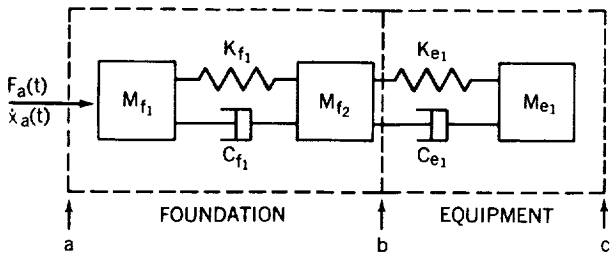



Figure 6—Lumped parameter representation of a one-dimensional foundation-equipment system.

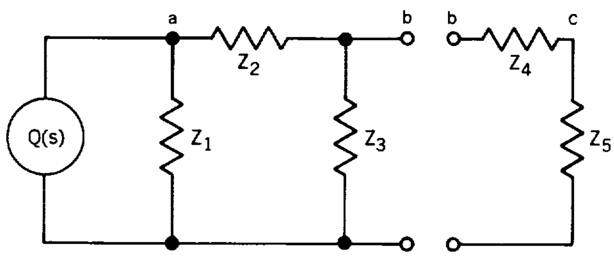



Figure 7—Mechanical network equivalent of Figure 6.

In Figure 7,

$$\left.\begin{aligned} Z_1 &= sM_{f_1} \\ Z_2 &= \frac{K_{f_1}}{s} + C_{f_1} \\ Z_3 &= sM_{f_2} \\ Z_4 &= \frac{K_{e_1}}{s} + C_{e_1} \\ Z_5 &= sM_{e_1} \end{aligned}\right\} \tag{d}$$

Block terminal b and determine the transform blocked force $Q_b^b$ corresponding to the blocked force $F_b^b(t)$ exerted at that terminal. Assuming the excitation

$$F_a(t) = F_0 \sin \omega t \tag{e}$$

and letting $Q(s)$ be the Laplace transform of $F_a(t)$, the transform $Q_b^b$ is

$$Q_b^b(s) = \left[1 + Z_1 Z_2^{-1}\right]^{-1} Q(s) . \tag{f}$$

(For other excitations, the appropriate expressions must be written.) Assuming that the impedance of the force generator is low ($\approx 0$), the impedance looking back into the foundation from the terminal b is

$$Z_f(s) = Z_1 Z_2 (Z_1 + Z_2)^{-1} + Z_3 ; \tag{g}$$

and the impedance looking forward into the equipment is

$$Z_e(s) = Z_4 Z_5 (Z_4 + Z_5)^{-1} . \tag{h}$$

1. Steady-state case:

Using Equation 3 and letting

$$H_1(s) = \left[Z_f(s) + Z_e(s)\right]^{-1}, \tag{i}$$

the transform velocity $\dot{q}_b$ corresponding to the velocity $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

$$\dot{q}_b(s) = H_1(s)\left[1 + Z_1 Z_2^{-1}\right]^{-1} Q(s) . \tag{j}$$

2. Transient case:

Determining the Fourier spectrum $G_{F_b^b}(i\omega)$ of the blocked force $F_b^b(t)$ from

$$G_{F_b^b}(i\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_b^b(t)\, e^{-i\omega t}\, dt, \tag{k}$$

where

$$F_b^b(t) = \int_{\omega=-\infty}^{+\infty} G_{F_a}(i\omega)\left[1 + Z_1(i\omega)\, Z_2^{-1}(i\omega)\right]^{-1} e^{i\omega t}\, d\omega , \tag{l}$$

and using Equation 50, the response $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

$$\dot{x}_b(t) = \int_{\omega=-\infty}^{\infty} H_1(i\omega)\, G_{F_b^b}(i\omega)\, e^{i\omega t}\, d\omega , \tag{m}$$

where

$$H_1(i\omega) = H_1(s)\Big|_{s=i\omega} . \tag{n}$$

3. Random case:

Determining the spectral density $S_{F_b^b}(\omega)$ of the blocked force $F_b^b(t)$ from

$$S_{F_b^b}(\omega) = \left|1 + Z_1(i\omega)\, Z_2^{-1}(i\omega)\right|^{-2} S_{F_a}(\omega) , \tag{o}$$

where $S_{F_a}(\omega)$ is the spectral density of the excitation $F_a(t)$, and using Equation 63, the spectral density $S_{\dot{x}_b}(\omega)$ of the response $\dot{x}_b(t)$ at the terminal b after connecting $Z_e$ is

$$S_{\dot{x}_b}(\omega) = \left|H_1(i\omega)\right|^2 \left|1 + Z_1(i\omega)\, Z_2^{-1}(i\omega)\right|^{-2} S_{F_a}(\omega) . \qquad \text{(p)}$$

The examples that would illustrate the application of the other equations of the section, "Impedance Analyses of One-Dimensional Systems," would be similar to the preceding examples.

## Reshaping of Frequency Response Characteristic of Test Machines

The method as proposed here involves the control of test-machine bare-table frequency response characteristics, so that appropriate mechanical impedance compensations are introduced: This results in dynamic characteristics at the table that are simulations of the vehicles' terminal characteristics. In some instances this control may be accomplished by various spectrum-shaping techniques employing present-day machine system equalization methods. It is the purpose here to present some general ideas behind such an approach.

In the section on impedance analyses, it was shown that the *free velocity* (bare-table velocity) of a test machine may be related to the *free velocity* or *blocked force*—as the case may be—of a vehicle. (Refer to Equations 21, 23, 57, 58, 75, and 76.) For this discussion, the steady-state case only shall be used, as the general idea is equally valid for the cases of other types of excitations. It can be shown from test-machine technology that the *blocked force* $Q^b_{S_m}(s)$ at the bare table may be related to the armature current $I_d(s)$ by an expression of the form

$$Q^b_{S_m}(s) = B_1(s)\, I_d(s) , \qquad (81)$$

where $B_1(s)$ is a transfer function. As the consequence of Equation 81, the *free velocity* of the test machine $\dot{q}_{S_m 0}(s)$ may be related by an expression in a similar form:

$$\dot{q}_{S_m 0}(s) = B_2(s)\, I_d(s) , \qquad (82)$$

where $B_2(s)$ may be considered as another transfer function. With the assumption that the above relations are practical, the nature of the control will be studied.

Let us suppose that an uncontrolled frequency response function is defined by

$$\log\left|\frac{\dot{q}_{S_m 0}(i\omega)}{I_d(i\omega)}\right| = \log\left|B_2(i\omega)\right| . \qquad (83)$$

For simulation the *free velocity*, Equation 21, may be expressed by

$$\dot{q}_{s_m 0}(i\omega) \rightarrow \dot{q}'_{s_m 0}(i\omega) \qquad K(i\omega)\, \dot{q}_{s_m 0}(i\omega) \tag{84}$$

and the new armature current by

$$I'_d(i\omega) = C(i\omega)\, I_d(i\omega) \ , \tag{85}$$

where $K(i\omega)$ and $C(i\omega)$ may be considered as correction factors.

The substitution of Equations 84 and 85 in 83 yields

$$\log\left|\frac{\dot{q}'_{s_m 0}(i\omega)}{I'_d(i\omega)}\right| = \log\left|B_2(i\omega)\right| + \log\left|R(i\omega)\right| \ , \tag{86}$$

where $R(i\omega) = K(i\omega)/C(i\omega)$ .

It is seen that the controlled frequency response function of the test machine is given by Equation 86; the last term on the right of the equation may be considered the correction to be made. This correction may be attempted by either mechanical or electrical means. Mechanically, it would involve mechanical fixture designs; and, electrically, it probably would involve spectrum-shaping techniques.

## Electronic Computer Applications

Advances made in servo control theory and problem simulation with electronic computers offer some hope that methods of mechanical impedance simulation could revolve around the use of electronic computer techniques. In view of the multitude of physical applications employing computers for real-time servo control and as problem simulators for analysis, these approaches offer some promise. The general ideas behind these approaches may be summarized as follows.

In servo control applications the electronic computer forms a part of a larger system. It receives impedance and velocity information from other parts of the system and from the outside, processes this information, and—as a result—furnishes instructions to other parts. Since the timing of the computer operations is tied in with that of the system, the computer works in real time. The type of computer (analog or digital) to be used depends on the natural advantages to be derived. The role of the computer is to perform mathematical computations so as to convert stored information supplied from outside into velocity magnitudes to be used as variable inputs (level sets) to the servo control system.

For use as problem simulators for analysis, special-purpose computers may be built. For instance, if a mechanical system is defined by the appropriate impedances, the computer can simulate the reaction of the components of the system. If true dynamic inputs are employed, the computer outputs would yield realistic dynamic responses of the system.

The details of these approaches have not been investigated, but the authors believe that such approaches are possible and warrant further study.

## DISCUSSION

The objective of this paper was not to investigate in detail the many existing problems in employing the concepts of mechanical impedance to dynamic environmental simulation but rather to provide guidelines based on theoretical considerations, by which certain aspects of simulation possibly may be achieved.

The equations governing the nature of simulation for one-dimensional structural systems have been derived and are summarized in the section on impedance analyses. Study of these equations shows the potential errors resulting from neglect of mechanical impedance effects during measurements, testing, and design of a system. Too frequently the tacit assumption that shock and vibration testing machines possess dynamic characteristics much like those of the actual equipment foundation is not true. Likewise, the tacit assumption that foundation impedance is large in comparison with equipment impedance is often wrong and may result in large errors in design and subsequent qualification testing of the system.

The possible magnitude of the bad effects in neglecting mechanical impedance should stimulate a desire to understand further the mechanisms by which the dynamic characteristics of system components alter the behavior of combined system dynamic environments. Theoretically, the effect of impedance variations on the motional parameter at any joining terminal of system components is related both to the influence of the backward and forward impedance at the terminal and to the nature of the type of excitations. This point is illustrated by the equations summarized in the impedance analyses section.

In the opinion of the authors, some amount of structural impedance consideration in design and qualification procedures is highly necessary to prove the suitability and efficiency of any new type of structural design or to improve existing designs.

The application of mechanical impedance simulation to systems of more than one dimension is an exceedingly complex problem and requires much more study. However, some practices now used yield such large errors that even one-dimensional analysis (which often will be approximate) should yield significant improvement.

## CONCLUSIONS

Several main conclusions may be drawn from this study:

1. The study, and possibly the control, of pertinent structural dynamic characteristics for one-dimensional structures that are linear-passive in nature is theoretically feasible, by use of mechanical impedance.

2. The usefulness of these mechanical impedance relations is of three kinds: (1) as a means of expressing dynamic characteristics of component structures in a manner that concisely tells the experienced engineer what he needs to know about the structures; (2) as a description of integral parts of structures, since it can be used together with a comparable description of the dynamic inputs to specify the response at any desired terminal; and (3) if dynamic responses are known for a particular integrated structure, the modified responses when integral parts of the structure are altered can be determined provided impedance information for the new and old integrated structures are known.

3. The practical applications of the mechanical impedance relations covered in the foregoing theory cannot be stated accurately because of the restrictive assumptions underlying this work. However, the proper extent of their application may be determined by suitable experiments.

## RECOMMENDATIONS

Several recommendations result from this investigation:

1. Experimental studies should be undertaken to validate (or refute) the usefulness of the theoretical relations summarized in this paper and to define limits of applicability and validity for each type of excitation.

2. Effort should be directed toward the development of sensors for measurement and the development of data reduction equipment and methods for analysis of the parameters that have been discussed.

3. The validity of the approaches presented to improve test control or simulation should be investigated in further studies.

4. As most structures are not ideal one-dimensional systems, effort should be expended toward developing theoretical relations for multi-dimensional systems.

## ACKNOWLEDGMENTS

The authors wish to express their sincere appreciation to Mr. William Forlifer and to Mr. Neal Granick of the Structural Dynamics Branch, Test and Evaluation Division, Goddard Space Flight Center, for their most helpful suggestions during the preparation of the manuscript.

## REFERENCES

1. Olson, H. F., "Dynamical Analogies," 2nd Ed., Princeton, N. J.: Van Nostrand, 1958.

2. von Kármán, T., and Biot, M. A., "Mathematical Methods in Engineering," New York: McGraw-Hill, 1940, pp. 365-409.

3. Blake, R. E., and Belsheim, R. O., "The Significance of Impedance in Shock and Vibration," in: *Colloquium on Mechanical Impedance Methods for Mechanical Vibrations*, ed. by R. Plunkett, New York: American Society of Mechanical Engineers, 1958, pp. 101-107.

4. Firestone, F. A., "The Mobility and Classical Impedance Analogies," in: *American Institute of Physics Handbook,* New York: McGraw-Hill, 1957, Section 3, pp. 140-177.

5. Lawson, J. L., and Uhlenbeck, G. E., "Threshold Signals," New York: McGraw-Hill, 1950.

Appendix A

# Symbols

## General

$a, k$ sub-subscripts for specific vehicle

$ab, ar, kb, mb$ denote respectively the joining terminal of $V_a - P_b$, $V_a - P_r$, $V_k - P_b$, and $S_m - P_b$

$b, r$ sub-subscripts for specific payload

$m$ sub-subscript for specific test machine

$N$ denotes $N$th harmonic component

$P$ subscript for payload

$S$ subscript for test machine

$s$ Laplace variable

$t$ time variable

$V$ subscript for vehicle

$\omega$ circular frequency

$\rightarrow$ denotes simulating a quantity

## Velocities

$\dot{q}(s)$ transform velocity

$\dot{q}_{V_{a}0}(s)$ transform *free velocity* of $V_a$ at terminal $a$

$\dot{q}_{V_{k}0}(s)$ transform *free velocity* of $V_k$ at terminal $a$

$\dot{q}_{S_{m}0}(s)$ transform *free velocity* of $S_m$ at terminal $a$

$\dot{q}_{V_{ab}}(s)$ transform velocity at joining terminal of $V_a - P_b$

$\dot{q}_{V_{ar}}(s)$ transform velocity at joining terminal of $V_a - P_r$

$\dot{q}_{V_{kb}}(s)$ transform velocity at joining terminal of $V_k - P_b$

$\dot{q}_{S_{mb}}(s)$ transform velocity at joining terminal of $S_m - P_b$

$G_{\dot{x}}(i\omega)$ Fourier spectrum of a transient velocity $\dot{x}(t)$

$S_{\dot{x}}(\omega)$ spectral density of a random velocity $\dot{x}(t)$

$\dot{x}(t)$ time-dependent velocity

## Forces

$F_N^b$ amplitude of Nth harmonic component *blocked force*

$G_F(i\omega)$ Fourier spectrum of a transient force $F(t)$

$Q(s)$ transform force

$Q_{V_a}^b(s)$ transform *blocked force* of $V_a$ at terminal a

$Q_{V_k}^b(s)$ transform *blocked force* of $V_k$ at terminal a

$S_F(\omega)$ spectral density of a random force $F(t)$

## Impedances

$Z(s) = Q(s)/\dot{q}(s)$, definition of transform mechanical impedance

$Z(i\omega) = Z(s)\big|_{s=i\omega}$

$Z_{P_b}(s)$ mechanical impedance of $P_b$ looking forward from terminal a

$Z_{P_r}(s)$ mechanical impedance of $P_r$ looking forward from terminal a

$Z_{S_m}(s)$ mechanical impedance of $S_m$ looking back from terminal a

$Z_{V_a}(s)$ mechanical impedance of $V_a$ looking back from terminal a

$Z_{V_{ab}}(s)$ point mechanical impedance at joining terminal of $V_a - P_b$

## Frequency Response Functions

$E_{1,6}(s) = H_1(s)H_6^{-1}(s)$

$E_{2,6}(s) = H_2(s)H_6^{-1}(s)$

$$E_{8,2}(i\omega) = H_8(i\omega)\, H_2^{-1}(i\omega)$$

$$E_{9,1}(i\omega) = H_9(i\omega)\, H_1^{-1}(i\omega)$$

$$E(i\omega) = E(s)\big|_{s=i\omega}$$

$$H(i\omega) = H(s)\big|_{s=i\omega}$$

$|H(i\omega)|$ = absolute value of $H(s)\big|_{s=i\omega}$

$$H(iN\omega) = H(s)\big|_{s=iN\omega}$$

$$H_1(s) = \left[Z_{V_a}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_2(s) = Z_{V_a}(s)\left[Z_{V_a}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_3(s) = \left[Z_{V_a}(s) + Z_{P_b}(s)\right]\left[Z_{V_a}(s) + Z_{P_r}(s)\right]^{-1}$$

$$H_4(s) = \left[\dot{q}_{V_k 0}(s)\right]\left[\dot{q}_{V_a 0}(s)\right]^{-1}\left[1 + Z_{P_b}(s)\, Z_{V_a}^{-1}(s)\right]\left[1 + Z_{P_b}(s)\, Z_{V_k}^{-1}(s)\right]^{-1}$$

$$H_5(s) = \left[Q_{V_k}^{b}(s)\right]\left[Q_{V_a}^{b}(s)\right]^{-1}\left[Z_{V_a}(s) + Z_{P_b}(s)\right]\left[Z_{V_k}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_6(s) = Z_{S_m}(s)\left[Z_{S_m}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_7(s) = \left[Z_{S_m}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_8(s) = Z_{V_k}(s)\left[Z_{V_k}(s) + Z_{P_b}(s)\right]^{-1}$$

$$H_9(s) = \left[Z_{V_k}(s) + Z_{P_b}(s)\right]^{-1}$$